Disordered Transverse Field Ising Chain

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Motivation

Quantum disordered systems have rich and unique phenomena, in which the pure system does not have.

These novel properties include Griffiths phase, Spin-Glass order, to Many-Body Localizations. But in general, it is very hard to solve them numerically, due to the exponential time complexity.

Transverse field Ising Chain is a prototype model for quantum phase transitions. By Jordan-Wigner transformation, we can map it to a free fermion problem. And solve it exactly.

Even with longer range interactions $\sigma_i^x\sigma_{i+1}^z\sigma_{i+2}^x$ added, the problem is still a free fermion problem. Therefore, a lot of theoretical questions can be answered in this framework.

Methods

There is a whole class of transverse field Ising models, which can be mapped to quadratic fermion or Majorana representation, by Jordan-Wigner transformations.

$$\sigma_i^z = ib_ia_i$$
$$\sigma_i^x\sigma_{i+1}^x = ia_{i+1}b_i$$
$$\sigma_i^x\sigma_{i+1}^z\sigma_{i+2}^x = ia_{i+2}b_i$$
$$\sigma_i^x\sigma_{i+1}^z\sigma_{i+2}^z\sigma_{i+3}^x = ia_{i+3}b_i$$
$$\cdots$$
$$\sigma_i^y\sigma_{i+1}^y = ib_{i+1}a_i$$
$$\sigma_i^y\sigma_{i+1}^z\sigma_{i+2}^y = ib_{i+2}a_i$$
$$\sigma_i^y\sigma_{i+1}^z\sigma_{i+2}^z\sigma_{i+3}^y = ib_{i+3}a_i$$
$$\cdots$$

$$H = \Psi^\dagger \begin{pmatrix} A & B \\ -B & -A \end{pmatrix} \Psi$$

$$H = -\sum_{i=1}^{L} h_i\sigma_i^z - \sum_{i=1}^{L-1}\lambda_{1i}\sigma_i^x\sigma_{i+1}^x - \sum_{i=1}^{L-2}\lambda_{1i}\sigma_i^x\sigma_{i+1}^z\sigma_{i+2}^x - \cdots$$

After solving them in the free fermion picture. Any physical quantity in spin picture can be calculated by inverse J-W transformation. This involves the structure called Pfaffian.

Pfaffian is the wick's theorem applied to anti-commute fermions

$$\langle \sigma_i^x\sigma_j^x\rangle = \langle a_1b_1a_2b_2\cdots a_ia_1b_1a_2b_2\cdots a_j\rangle$$

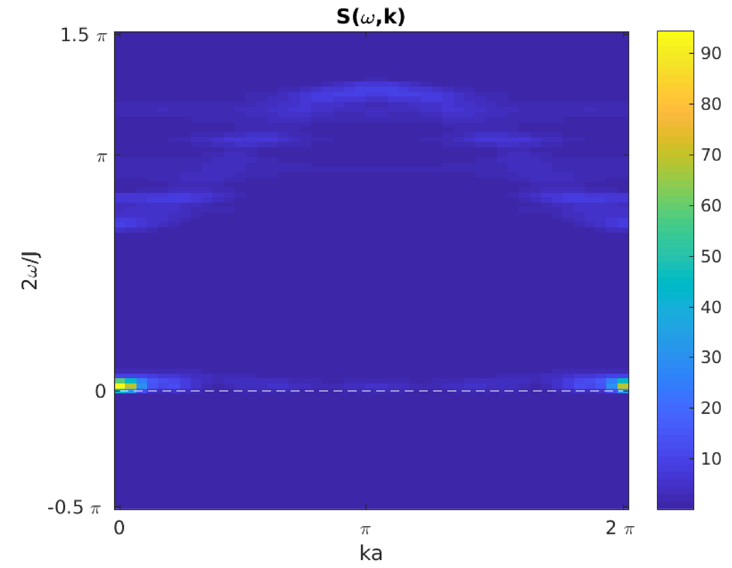
$$\langle ABC\cdots\rangle = \mathrm{Pf}\begin{bmatrix} 0 & \langle AB\rangle & \langle AC\rangle & \\ -\langle AB\rangle & 0 & \langle BC\rangle & \\ -\langle AC\rangle & -\langle BC\rangle & 0 & \\ & & & \ddots \end{bmatrix}$$
$$= \langle AB\rangle\langle CD\rangle\cdots - \langle AC\rangle\langle BD\rangle\cdots + \cdots$$

Emergent low energy modes in disorder system

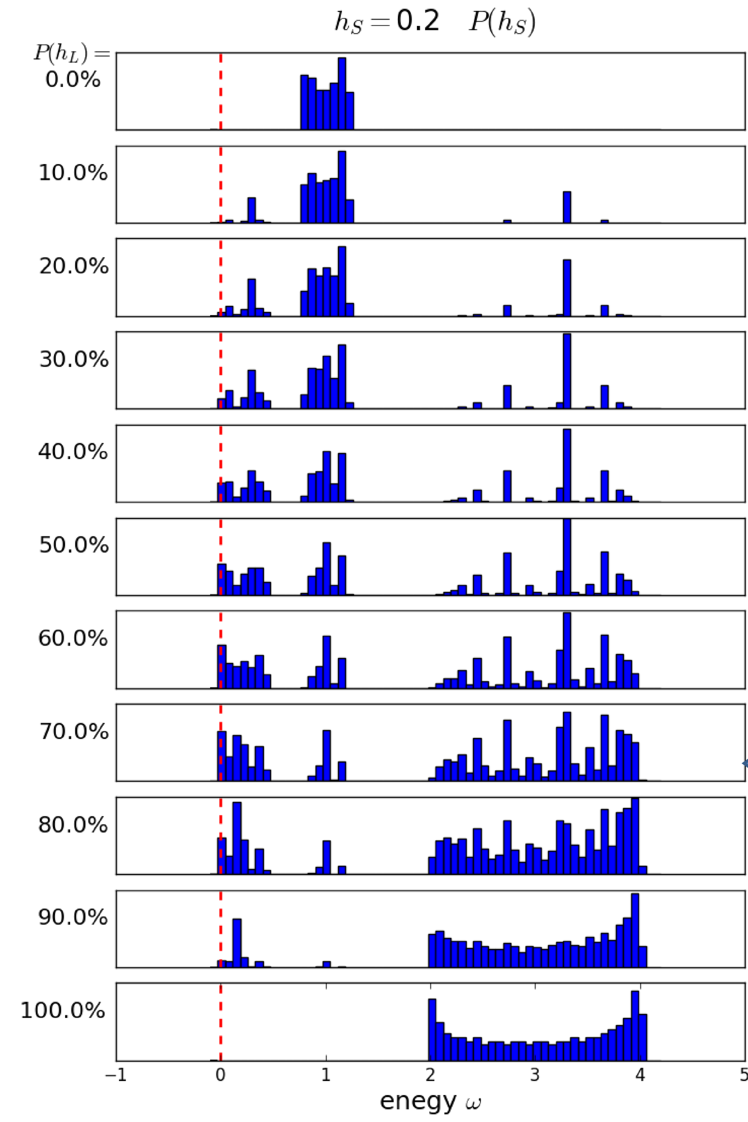
$$H = -\sum_{i=1}^{L} h_i\sigma_i^z - \sum_{i=1}^{L-1}\sigma_i^x\sigma_{i+1}^x$$

$$h_i = \begin{cases} 0.2 & 40\% \\ 3.0 & 60\% \end{cases}$$

$$S(\omega,k) = \mathrm{FourierTransform}[\overline{\langle \sigma_i^x(t)\sigma_j^x\rangle}]$$



Dynamical structure factor $S(\omega,k)$ can be directly detected by neutron scattering experiment.
We observed **emergent low energy models** in the binary disorder system.
This won't happen in any pure systems.



Density of state plot for binary disorders. P is like doping percentage.
$P = 0\%$ corresponds to quantum ferromagnetic phase
$P = 100\%$ corresponds to quantum paramagnetic phase

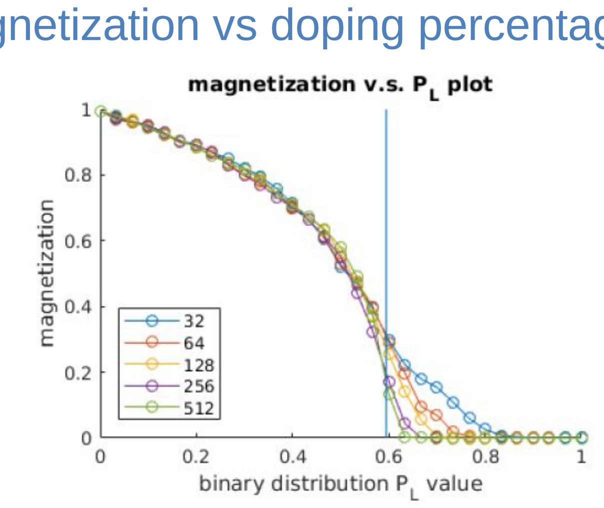
$$H = -\sum_{i=1}^{L} h_i\sigma_i^z - \sum_{i=1}^{L-1}\sigma_i^x\sigma_{i+1}^x$$

$$h_i = \begin{cases} 0.2 & 1-P \\ 3.0 & P \end{cases}$$

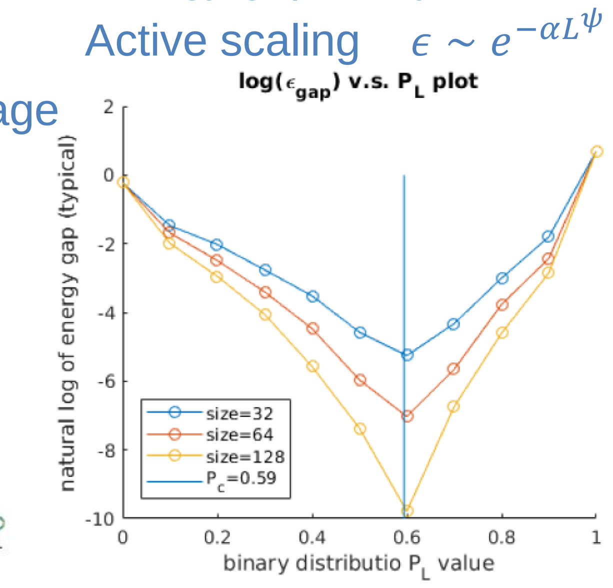
$$P_C = \frac{\ln h_S}{\ln h_S - \ln h_L}$$

Infinite randomness fixed point

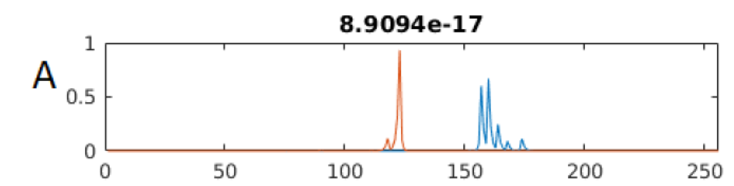
Magnetization vs doping percentage



Energy gap vs System size
Active scaling $\epsilon \sim e^{-\alpha L^\psi}$



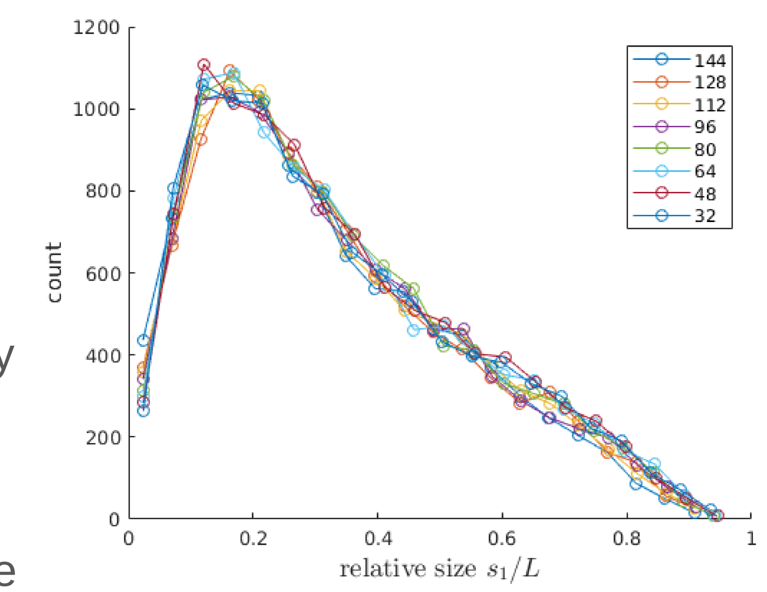
Disorder induced rare regions Majorana zero modes within the bulk



The rare regions in disorder transverse field Ising model, will support Majorana modes at their boundaries.

$$s_n = \left|\frac{\sum_i i|\psi_{in}|^2}{\sum_i|\psi_{in}|^2} - \frac{\sum_i i|\phi_{in}|^2}{\sum_i|\phi_{in}|^2}\right|$$

The size of the Majorana modes can be well defined, and calculated.



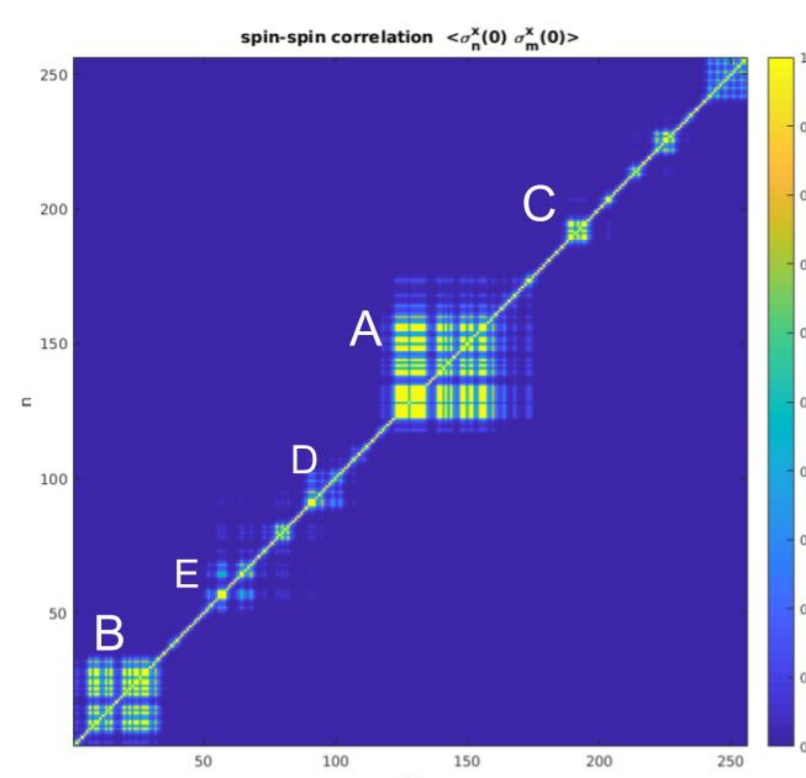
We find that:
right at the critical point, the relative size distribution of the largest Majorana modes, all collapses to the same curve, at all length scale.

Such self-similarity only happens at the critical point. Away from the critical point, the curves will flow.

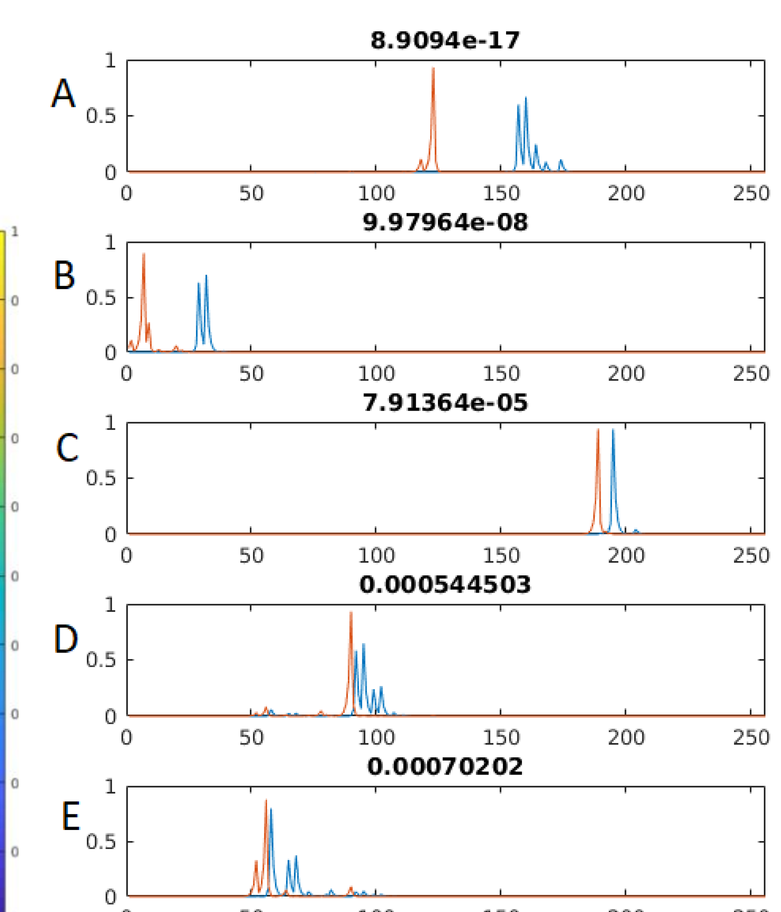
a Cantor set

The self similarity is a signature of fractal geometry.
In our disorder problem, it is a random Cantor set.
The lines are the magnetic rare regions,
and the end points are a pair of Majorana zero modes.

Griffith rare magnetic regions



Majorana zero mode pairs

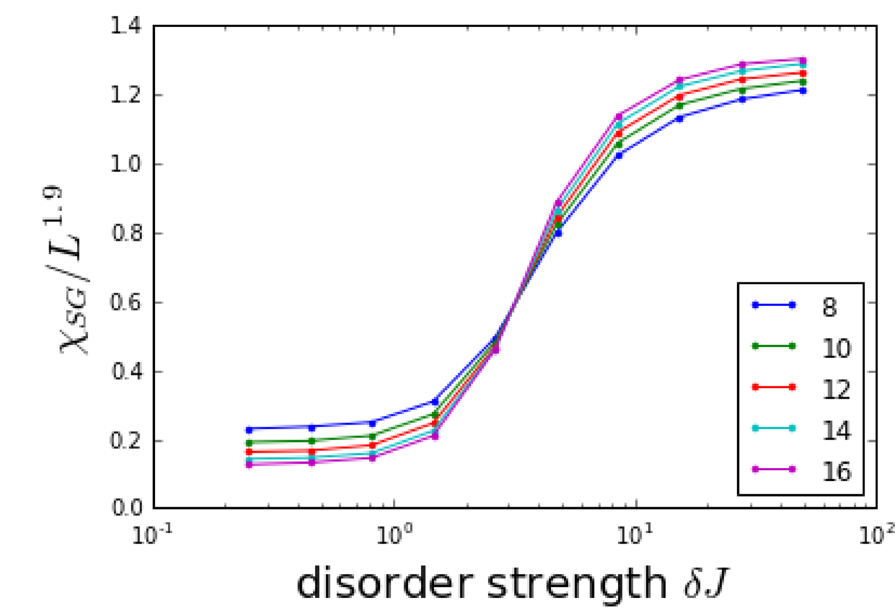


Spin Glass Susceptibility

Like most spin systems, by introducing frustration and disorder, we can observe the spin glass transitions.

$$\chi_{SG} = \sum_{i,j=1}^{L}\overline{\langle\sigma_i^x\sigma_j^x\rangle^2}$$

$h = 1$
$\lambda_2 = -0.3$ (frustrations)
$\lambda_{1i} \sim \mathrm{Uniform\ Distribution}[\lambda_1 - \delta J, \lambda_1 + \delta J]$
where $\lambda_1 = 1$ (disorder)



Conclusion

We've shown that, quenched disorder can induce rare regions, and there will be zero energy Majorana modes at the boundaries of the rare regions.
The existing condition of low energy mode is, the disorder parameters have to take values from different phases.
The quality(gap closing and number percentage) of the low energy mode will reach a peak value at the infinite randomness fixed point. The separation distance distribution of lowest energy Majorana mode pairs, is defined.
This quantity is very easy to calculate, and it will have fractal behavior at the critical point.
The whole calculation can be done up to 200 sites, with no exponential time complexity.

References

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